## §3. Isomorphic Groups and Homomorphic Mappings.

(A). Let $G$ and $\Gamma$ be groups and let $f$ be a mapping of $G$ into $\Gamma$, or (what it the same thing) a function whose argument ranges through $G$ and whose values lie in $\Gamma$. If

$$f(\xi\eta) = f(\xi)f(\eta) \qquad (1)$$

for all $\xi, \eta$ in $G$, then $f$ is called homomorphic; it is also called a homomorphism of $G$ into $\Gamma$.

Taking $\eta = 1$, we find that, if $f$ is homomorphic, then $f(1)$ is the unit element of $\Gamma$. Taking $\eta = \xi^{-1}$, we deduce that

$$f(\xi^{-1}) = f(\xi)^{-1}. \qquad (2)$$

The image of $G$ under $f$ is the set $f(G)$ of all $f(\xi)$ with $\xi \in G$. $f(G)$ is a subgroup of $\Gamma$. If $f(G) = \Gamma$, then $f$ is said to map $G$ onto $\Gamma$: it is an epimorphism. On the other hand, if $f(\xi) = f(\eta)$ implies $\xi = \eta$, then $f$ is a one-to-one mapping: it is a monomorphism. If $f$ is both an epimorphism and a monomorphism, then it is called as isomorphism of $G$ onto $\Gamma$. In this case, the inverse mapping $f^{-1}$ exists and this will be an isomorphism of $\Gamma$ onto $G$.

Two groups $G$ and $\Gamma$ are called isomorphic if there exists an isomorphism $f$ mapping one onto the other. This relation is written

$$G \cong \Gamma.$$

The isomorphism $f$ allows us to translate any group-theoretical statement about $G$ into a corresponding statement about $\Gamma$. Conversely, using $f^{-1}$. Hence isomorphic groups are simply copies of each other. They have the same order and their subgroups can be put in one-to-one correspondence, with normal subgroups corresponding to normal subgroups, joins and intersections being preserved and so on. We may describe them as instances of the same type of group. For example, cyclic groups of the same order are isomorphic. If $|X| = |Y|$, then $\Sigma(X) \cong \Sigma(Y)$.

Evidently, a monomorphism $f$ of $G$ into $\Gamma$ is at the same time an isomorphism of $G$ onto $f(G)$, which is a subgroup of $\Gamma$.

From 1·4 we have

Theorem 3·1. For any group $G$, the mapping
$$\xi \to r(\xi) \qquad (\xi \in G)$$
is a monomorphism of $G$ into $\Sigma(G)$, called the regular representation of $G$. Thus $G \cong r(G)$; every group is isomorphic with a permutation group.

This remark is due to A. Cayley 1821–95. Note that the mapping
$$\xi \to l(\xi^{-1}) \qquad (\xi \in G)$$
is also a monomorphism of $G$ into $\Sigma(G)$.

Another example follows from 2·8. This is

Theorem 3·2 Let $H \triangleleft G$. Then the mapping
$$\xi \to H\xi \qquad (\xi \in G)$$
is an epimorphism of $G$ onto $G/H$. This is called the natural epimorphism

(B). Every homomorphism can be analysed into a natural epimorphism followed by a monomorphism. ~~Fi~~ If $G$ and $\Gamma$ are groups, let
$$\mathrm{Hom}(G, \Gamma)$$
denote the set of all possible homomorphisms of $G$ into $\Gamma$. Suppose that $f \in \mathrm{Hom}(G,\Gamma)$, and let $f(\xi) = f(\eta)$. Then $f(\xi\eta^{-1}) = 1$, the unit element of $\Gamma$, by (1) and (2). Conversely $f(\xi\eta^{-1}) = 1$ implies $f(\xi) = f(\eta)$. The set $K$ of all $\xi \in G$ such that $f(\xi) = 1$ is called the kernel of $f$. For any $\eta \in G$ and $\xi \in K$, we have $f(\eta^{-1}\xi\eta) = f(\eta)^{-1} f(\xi) f(\eta) = 1$ since $f(\xi) = 1$. Hence $\eta^{-1}K\eta = K$ or $K\eta = \eta K$ for all $\eta \in G$. Since $K$ is clearly a subgroup of $G$, it follows that $K \triangleleft G$. And $f(\xi) = f(\eta)$ if and only if $\xi$ and $\eta$ belong to the same coset of $K$. Hence we have a one-to-one mapping
$$K\xi \to f(\xi)$$
of the cosets of $K$ onto the elements of $f(G)$. Since $K$ is normal in $G$

$K\xi \, K\eta = K\xi\eta \rightarrow f(\xi)f(\eta) = f(\xi\eta)$. So this mapping is an isomorphism of $G/K$ onto $f(G)$. This gives

<u>Theorem 3·3</u>. If $f$ is any homomorphism of $G$ with kernel $K$, then $K \triangleleft G$ and $G/K \cong f(G)$.

Thus the mapping $\xi \rightarrow f(\xi)$, $\xi \in G$, is the product of the natural epimorphism $\xi \rightarrow K\xi$ of $G$ onto $G/K$, followed by the isomorphism $K\xi \rightarrow f(\xi)$ of $G/K$ onto $f(G)$.

<u>Theorem 3·4</u> Let $H$ be any subgroup of $G$ and let $K \triangleleft G$. Then $KH$ is a subgroup of $G$ and $K \cap H \triangleleft H$ and

$$H/K\cap H \cong KH/K.$$

Proof: since $K \triangleleft G$, it is permutable with $H$. So $KH$ is a subgroup of $G$ by 2·7. Obviously $K \triangleleft KH$. Let $f$ be the restriction to $H$ of the natural epimorphism of $G$ onto $G/K$. Then $f(H) = KH/K$. Since $f$ is a homomorphism of $H$ with kernel $K \cap H$, the result follows from 3·3. It is usually called the <u>first isomorphism theorem</u>.

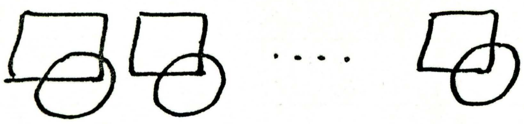
(C). If $K \triangleleft G$ and $L$ is any subgroup of $G$ containing $K$, then $K \triangleleft L$ and $L/K$ is a subgroup of $G/K$. Every subgroup of $G/K$ is of this form. Moreover $L \triangleleft G$ if and only if $L/K \triangleleft G/K$. Now if $L \triangleleft G$, the product $f$ of the natural epimorphisms of $G$ onto $G/K$ and of $G/K$ onto $\Gamma = (G/K)/(L/K)$ maps any element $\xi$ of $G$ onto that element $f(\xi)$ of $\Gamma$ which contains $K\xi$ as as one of its members. Thus $f(\xi)$ is simply $L\xi$ considered not as a set of elements of $G$ but as a union of certain cosets of $K$. Comparing $f$ with the natural epimorphism of $G$ onto $G/L$ we obtain

<u>Theorem 3·5</u> Let $K$ and $L$ be normal subgroups of $G$ such that $L$ contains $K$. Then

$$G/L \cong (G/K)/(L/K).$$

This rather obvious fact is sometimes known as the <u>second isomorphism theorem</u>.

(D). Let $H/K$ and $L/M$ be two sections of the group $G$. There is one simple case where we can be sure that $H/K \cong L/M$. This is when each element $K\eta$ $(\eta \in H)$ of $H/K$ "meets" (has non-empty intersection with) exactly one element $M\lambda$ $(\lambda \in L)$ of $L/M$ and when conversely each element of $L/M$ meets exactly one element of $H/K$. The situation is like this:



where for example the squares represent elements of $H/K$ and the circles elements of $L/M$. Two sections related in this way are called <u>incident</u>. If we make each $K\eta$ $(\eta \in H)$ correspond to the unique $M\lambda$ $(\lambda \in L)$ which it meets, we obtain an isomorphism of $H/K$ onto $L/M$. Thus incident sections are isomorphic.

Now let $H/K$ and $L/M$ be any two sections of $G$. By 3·4, $L^* = K(L \cap H)$ is a subgroup of $H$ because $K \lhd H$. Similarly $M^* = K(M \cap H)$ is a subgroup and $K \leq M^* \leq L^* \leq H$. By 3·4 again, $M \cap H \lhd L \cap H$ because $M \lhd L$. Hence if $\eta \in L \cap H$ we have $\eta K = K\eta$ and $\eta(M \cap H) = (M \cap H)\eta$. Consequently $\eta M^* = M^*\eta$. But $K \leq M^*$ and so every element of $K$ is also permutable with $M^*$. It follows that every element of $L^* = K(L \cap H)$ is permutable with $M^*$. Consequently $M^* \lhd L^*$.

We call the quotient group $L^*/M^*$ the <u>projection</u> of $L/M$ in $H/K$. ~~In view of 3.5, it may be regarded as a section of H/K.~~

Similarly we can form the projection $H^\dagger/K^\dagger$ of $H/K$ in $L/M$. Here $H^\dagger = M(H \cap L)$ and $K^\dagger = M(K \cap L)$. Then we have the Zassenhaus Lemma:

<u>Theorem 3·6</u> If $H/K$ and $L/M$ are any two sections of a group, then their projections $L^*/M^*$ and $H^\dagger/K^\dagger$ in each other are incident. Hence we have

$$K(L \cap H)/K(M \cap H) \cong M(H \cap L)/M(K \cap L).$$

Proof: Since $K \leq M^* \leq L^*$, we have $L^* = M^*(L \cap H)$ and every element of $L^*/M^*$ has the form $M^*\lambda$ with $\lambda \in L \cap H$. Similarly every element of $H^\dagger/K^\dagger$ has the form $K^\dagger\lambda$ with $\lambda \in L \cap H$. Hence each element of $L^*/M^*$ "meets" some element of $H^\dagger/K^\dagger$ and vice versa.

Suppose $K^\dagger\lambda$ meets both $M^*\lambda_1$ and $M^*\lambda_2$ where $\lambda$, $\lambda_1$ and $\lambda_2$ are in $L \cap H$. Since $M \triangleleft L$, we have $K^\dagger = M(L \cap K) = (L \cap K)M$ and so $\kappa_1\mu_1\lambda \in M^*\lambda_1$ and $\kappa_2\mu_2\lambda \in M^*\lambda_2$ for some $\kappa_1, \kappa_2$ in $L \cap K$ and some $\mu_1, \mu_2$ in $M$. Hence $(\kappa_1\mu_1\lambda)(\kappa_2\mu_2\lambda)^{-1} = \kappa_1\mu_1\mu_2^{-1}\kappa_2^{-1} \in L^* \leq H$ and so $\mu_1\mu_2^{-1} \in M \cap H$. Since $K \triangleleft H$, it follows that $\kappa_1\mu_1\mu_2^{-1}\kappa_2^{-1} \in K(M \cap H) = M^*$. Hence $M^*\lambda_1 = M^*\lambda_2$ and $K^\dagger\lambda$ meets only one element of $L^*/M^*$. Similarly each element of $L^*/M^*$ meets only one element of $H^\dagger/K^\dagger$. Thus the two projections are incident, as stated.

The first isomorphism theorem is a special case of the Zassenhaus Lemma, viz. the case $H = G$, $M = 1$.

(E). Let

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_r = G. \qquad (1)$$

Such a chain of subgroups is called a series of $G$; each term is normal in the next. Let

$$1 = L_0 \triangleleft L_1 \triangleleft \cdots \triangleleft L_s = G \qquad (2)$$

be any other series, also going from 1 to $G$. Let

$$L_{ij} = H_{j-1}(L_i \cap H_j) \quad \text{and} \quad H_{ji} = L_{i-1}(H_j \cap L_i).$$

Then we obtain two new series:

$$1 = L_{01} \triangleleft L_{11} \triangleleft \cdots \triangleleft L_{s1} = H_1 = L_{02} \triangleleft L_{12} \triangleleft \cdots \triangleleft L_{s2} = H_2 = L_{03} \triangleleft \cdots$$
$$\cdots \triangleleft L_{s,r-1} = H_{r-1} = L_{0r} \triangleleft L_{1r} \triangleleft \cdots \triangleleft L_{sr} = H_r = G \qquad (3)$$

and $1 = H_{01} \triangleleft H_{11} \triangleleft \cdots \triangleleft H_{r1} = L_1 = H_{02} \triangleleft \cdots \triangleleft H_{rs} = L_s = G$. (4)

Since each term of (1) occurs also in (3), we call (3) a <u>refinement</u> of (1). It is obtained from (1) by the insertion of more terms. Similarly (4) is a refinement of (2).

The number $r$ is called the <u>length</u> of the series (1) and the

$r$ groups $H_j/H_{j-1}$ $(j=1,2,\dots,r)$ are called <u>factor groups</u> of $G$ : they are the <u>factors</u> of the series (1). The series (2) is of length $s$ and its factors are the $s$ groups $L_i/L_{i-1}$ $(i=1,2,\dots,s)$. The two series (3) and (4) have the same length $rs$. Now the factor $L_{ij}/L_{i-1,j}$ of (3) is the projection of $L_i/L_{i-1}$ in $H_j/H_{j-1}$ ; and the factor $H_{ji}/H_{j-1,i}$ is the projection of $H_j/H_{j-1}$ in $L_i/L_{i-1}$. Hence

$$L_{ij}/L_{i-1,j} \cong H_{ji}/H_{j-1,i}$$

and indeed, by 3·6, these two factor groups of $G$ are incident for all $i=1,2,\dots,s$ and all $j=1,2,\dots,r$. Thus we have

<u>Theorem 3·7</u> Any two series of $G$ possess refinements which are of equal length and with factors which are incident in pairs.

This is known as the <u>refinement theorem</u> : O. Schreier 1901–26.

(F). If $H_{i-1}=H_i$ in (1), the factor $H_i/H_{i-1}$ is called <u>trivial</u>. By omitting repeated terms, we can replace any given series by one which has the same non-trivial factors but has no trivial factors. A series without trivial factors, i.e. without repeated terms, is called <u>proper</u>. A <u>composition series</u> of $G$ is a proper series which has no proper refinements distinct from itself. By 3·5, a composition series may also be defined as one in which all the factors are simple groups. Obviously every proper series can be refined to a composition series. If we apply 3·7 to two given composition series of $G$ and then detrivialize the two refinements, we obtain

<u>Theorem 3·8</u> The factors of any two composition series of $G$ are incident in pairs.

This is the <u>Jordan-Hölder Theorem</u> : C. Jordan 1838–1922 and O. Hölder 1859–1937.

A <u>soluble</u> group is one which has a series with all the factors Abelian. Since subgroups and quotient groups of Abelian groups are always Abelian, the <u>composition factors</u> of a soluble groups, i.e. the factors of any composition series, are also Abelian. But all subgroups of an Abelian group are normal, and so by 2·5 the only simple Abelian groups are the groups of order a prime. Hence a group G is soluble if and only if all its composition factors are of prime order. Insoluble groups are distinguished by the fact that one at least of their composition factors is a simple group of composite order.