§12 Wreath Products, Burnside's Transfer Theorem, Z-groups, A-groups, supersolv…

(A) Let H and K be subgroups of $\Sigma(X)$ and $\Sigma(Y)$ respectively, and let $Z = X \times Y$ be the set of all ordered pairs (x, y) with $x \in X$, $y \in Y$. For $\xi \in H$, $y \in Y$ and $\eta \in K$ define the permutations ξ_y and $\bar{\eta}$ of Z by

$$(x, y)\xi_y = (x\xi, y) \; ; \quad (x, y_1)\xi_y = (x, y_1) \text{ if } y_1 \neq y \; ;$$
$$(x, y)\bar{\eta} = (x, y\eta) .$$

The group $H \wr K$ generated by all these permutations is called the <u>wreath product</u> of H with K. It is usual to identify $\bar{\eta}$ with η, so that K becomes a subgroup of $W = H \wr K$. For fixed $y \in Y$, the mapping $\xi \to \xi_y$ $(\xi \in H)$ is an isomorphism of H onto a subgroup H_y of W, and the product

$$\bar{H} = \prod_{y \in Y} H_y$$

is direct. $\bar{H}$ is called the <u>base group</u> of W. K is faithfully represented by automorphisms of $\bar{H}$ according to the law

$$\eta^{-1}\xi_y\eta = \xi_{y\eta} \qquad (\eta \in K, y \in Y, \xi \in H),$$

i.e. K permutes the $|Y|$ direct factors of $\bar{H}$. If this representation of K is f, then

$$H \wr K = \langle K, \bar{H} ; f \rangle$$

is the corresponding split extension. $W = \bar{H}K$; $\bar{H} \cap K = 1$; $\bar{H} \lhd K$. If $|X| = l$, $|Y| = m$, then W is a permutation group of degree lm and order $|H|^m \cdot |K|$.

If L is a subgroup of $\Sigma(V)$, where $|V| = n$ and if we make the natural identification $((x, y), v) = (x, (y, v)) = (x, y, v)$, then the two groups $(H \wr K) \wr L$ and $H \wr (K \wr L)$ coincide. This is <u>the associative law for wreath products</u>, and permits us to omit brackets in a repeated wreath product.

If H or K or both are not given explicitly as permutation groups, then $H \wr K$ is to be interpreted as being formed from the appropriate regular representation. We call this the regularity convention. For example we have

Lemma 12·1 If C is a cyclic group of order p, then the group

$$\wr^n C = \underbrace{C \wr C \wr \cdots \wr C}_{n}$$

is a Sylow p-subgroup of $\Sigma(C^n)$.

For $|C^n| = p^n$, C^n being the set of all ordered n-tuplets of elements of
Hence $|\Sigma(C^n)|_p = (p^n!)_p = p^{1+p+p^2+\cdots+p^{n-1}} = |\wr^n C|$ by induction on n.
This remark is due to Kaloujnine.

Note that $H \wr K$ is determined to within isomorphism by the group H and the permutation group K, and does not depend as a group on the permutational representation chosen for H.

(β) Lemma 12·2 If H and K are given groups and $W = H \wr K$ (with the regularity convention), then every subgroup K_1 of W which is transversal to the base group $\overline{H}$ in W is conjugate to K in W.

Proof: Here we may identify $\xi \in H$ with $\xi_1 \in H_1$, so that $H = H_1$ and $H_\alpha = H^\alpha$, $\xi_\alpha = \alpha^{-1}\xi\alpha = \xi^\alpha$ for $\alpha \in K$. $\overline{H}$ is the direct produ of the H^α, $\alpha \in K$. Every element $u \in \overline{H}$ is uniquely expressible as a product $\prod_{\alpha \in K} u_\alpha{}^\alpha$, with factors $u_\alpha \in H = H_1$. Since K_1 is transversa to $\overline{H}$ in W, the coset $\alpha\overline{H}$ contains exactly one element $\alpha u(\alpha)$ of K and since K_1 is a subgroup of W, we have $u(\alpha)^\beta u(\beta) = u(\alpha\beta)$ for all $\alpha, \beta \in K$. Factorising, this gives

$$u(\alpha\beta)_\eta = u(\alpha)_{\eta\beta^{-1}}\, u(\beta)_\eta \qquad (1)$$

for all $\alpha, \beta, \eta \in K$. Let $v \in \overline{H}$ have factors $v_\beta = u(\beta^{-1})_1$ ~~...~~
so that $v = \prod_{\beta \in K} (u(\beta^{-1})_1)^\beta$. The β-factor of v^α is therefore $v_{\beta\alpha^{-1}}$
$= u(\beta\alpha^{-1})_1$. But the β-factor of $u(\alpha)v$ is $u(\alpha)_\beta\, u(\beta^{-1})_1 = u(\alpha\beta^{-1})_1$
by (1), and this is precisely the β-factor of $v^{\beta\alpha}$. Hence $v^\alpha = u(\alpha)v$ for all $\alpha \in K$, and so ~~...~~ $\alpha u(\alpha) = v\alpha v^{-1}$. Thus $K = K_1^v$ is conjuga to K_1 in W as stated.

(C) <u>Lemma 12·3</u> Let H be a subgroup of G and let S be any transversal to H in G, with elements $\sigma_1, \sigma_2, \dots, \sigma_m$ where $m = |G:H|$. For $\xi \in G$ and $1 \le i \le m$, define $i\xi$ by the condition $\sigma_i \xi \sigma_{i\xi}^{-1} \in H$, and let X be the set whose elements are the m cosets of H in G, so that $\tau_H(\xi)$ is the permutation $H\sigma_i \to H\sigma_i\xi = H\sigma_{i\xi}$ $(i = 1, \dots, m)$ of X. Define $m_H(\xi)$ to be the permutation

$$(\eta, H\sigma_i) \to (\eta\sigma_i\xi\sigma_{i\xi}^{-1}, H\sigma_{i\xi}) \qquad (\eta \in H, 1 \le i \le m)$$

of $H \times X$. Then m_H is an isomorphism mapping G into $H \wr \tau_H(G)$.

Here H is in its regular representation i.e. the regularity convention operates for H. Since $(i\xi)\xi_1 = i(\xi\xi_1)$ for $\xi, \xi_1 \in G$, we have $m_H(\xi\xi_1) = m_H(\xi)\, m_H(\xi_1)$. $m_H(\xi)$ is the identity on $H \times X$ only if $i\xi = i$ for all i and in addition $\sigma_i\xi\sigma_i^{-1} = 1$, which implies $\xi = 1$. Hence m_H is an isomorphic mapping of G into $\Sigma(H \times X)$. $W = H \wr \tau_H(G)$ contains the permutation $(\eta, H\sigma_i) \to (\eta, H\sigma_{i\xi})$ which we identify with $\tau_H(\xi)$, and $m_H(\xi)\,\tau_H(\xi)^{-1}$ maps $(\eta, H\sigma_i)$ into $(\eta\sigma_i\xi\sigma_{i\xi}^{-1}, H\sigma_i)$. This shows that $m_H(\xi)\,\tau_H(\xi)^{-1}$ belongs to the base group $\overline{H}$ of W and so $m_H(\xi) \in W = \overline{H}\,\tau_H(G)$. This proves 12·3.

It is clear that the representation m_H of G depends not only on the subgroup H byt also on the choice of the transversal S. However, preserving the notations of 12·3, we have

<u>Lemma 12·4</u> Let f be a homomorphism of the subgroup H of G into an Abelian group A, which we write in additive notation. Then the mapping f^* of G into A defined by

$$f^*(\xi) = \sum_{i=1}^{m} f(\sigma_i\xi\sigma_{i\xi}^{-1})$$

is also homomorphic and is independent of the choice of the transversal S. Moreover

$$f^*(\xi) = \sum_{j \in J} f(\sigma_j \xi^{r_j} \sigma_j^{-1})$$

where the cosets $H\sigma_j$ are chosen one from each cycle of $\tau_H(\xi)$, and r_j is the order of the cycle to which $H\sigma_j$ belongs, so that $\sum_{j \in J} r_j = |G:H| = m$.

Proof: Since f is homomorphic, we have for all i and all $\xi, \eta \in G$, $f(\sigma_i \xi \eta \sigma_{i\xi\eta}^{-1}) = f(\sigma_i \xi \sigma_{i\xi}^{-1}) + f(\sigma_{i\xi} \eta \sigma_{i\xi\eta}^{-1})$. Hence $f^*(\xi\eta) = f^*(\xi) + f^*(\eta)$ and f^* is homomorphic. If T is any other transversal to H in G, with elements $\tau_i = \eta_i \sigma_i$, $\eta_i \in H$, then $\tau_i \xi \tau_{i\xi}^{-1} = \eta_i (\sigma_i \xi \sigma_{i\xi}^{-1}) \eta_{i\xi}^{-1}$ and $f(\tau_i \xi \tau_{i\xi}^{-1}) = f(\eta_i) - f(\eta_{i\xi}) + f(\sigma_i \xi \sigma_{i\xi}^{-1})$. Summing for $i = 1, \dots, m$, we obtain $\sum_i f(\tau_i \xi \tau_{i\xi}^{-1}) = \sum_i f(\sigma_i \xi \sigma_{i\xi}^{-1})$ since $\sum_i f(\eta_i) = \sum_i f(\eta_{i\xi})$. Thus f^* is independent of the choice of transversal.

The contribution to $f^*(\xi)$ of the cycle of $\tau_H(\xi)$ which contains $H\sigma_j$ is $\sum_{i=0}^{r_j-1} f(\sigma_{j\xi^i} \xi \sigma_{j\xi^{i+1}}^{-1}) = f(\sigma_j \xi^{r_j} \sigma_j^{-1})$ since $j\xi^{r_j} = j$ and f is homomorphic. Thus 12.4 is proved

We call f^* the <u>transfer of f to</u> G. This is the terminology suggested by cohomology theory. However, if f is the natural homomorphism $\eta \to H'\eta$ $(\eta \in H)$ of H onto $A = H/H'$, then f^* is the <u>transfer of G into</u> H. It is a homomorphism of G into H/H' and, for $\xi \in G$, we have in this case

$$f^*(\xi) = H' \prod_{i=1}^{m} (\sigma_i \xi \sigma_{i\xi}^{-1}) = H' \prod_{j \in J} (\sigma_j \xi^{r_j} \sigma_j^{-1}). \qquad (1)$$

Note that $\sigma_j \xi^{r_j} \sigma_j^{-1}$ is the first positive power of $\sigma_j \xi \sigma_j^{-1}$ which lies in H

(D) <u>Theorem 12.5</u>. Let the group G have an Abelian Sylow p-subgroup let $N = N_G(H)$, $C = \mathfrak{z}N \cap H$, $D = [H, N]$. Let i be the identity mapping of H, i^* the transfer of i to G and K the kernel of i^*. Then H is the direct product of C and D. Also $C = i^*(G) = i^*(H)$ and $D = G' \cap H = K \cap H$. G is the split extension of K by C and $G/K \cong C$ is the maximal p-quotient group of G.

Proof: Since $H' = 1$, we have $i^*(\xi) = \prod_{j \in J} \sigma_j \xi^{r_j} \sigma_j^{-1}$ by (1) above, for any $\xi \in G$. If $\xi \in H$, then the elements ξ^{r_j} and $\sigma_j \xi^{r_j} \sigma_j^{-1}$ both lie in H and are conjugate in G. By 6.63, it follows that these two elements are conjugate in N, since $H = \mathfrak{z}H$ is Abelian. Since $\sum_{j \in J} r_j = m = |G:H|$, we may therefore write $\sigma_j \xi^{r_j} \sigma_j^{-1} = \tau_j^{-1} \xi^{r_j} \tau_j$

with $\tau_j \in N$, and so
$$i^*(\xi) = \xi^m \prod_{j\in J} [\xi^{r_j}, \tau_j] \qquad (2)$$
for all $\xi \in H$, where of course J, r_j and τ_j in general depend on ξ. If $\xi \in C$, then ξ^{r_j} commutes with τ_j and so $i^*(\xi) = \xi^m$. But C is a p-group and $(m, p) = 1$. Hence $i^*(C) = C$.

If $\eta \in N$ and $\xi \in G$, then $\eta\sigma_i\xi(\eta\sigma_j)^{-1} \in H$ if and only if $\sigma_i\xi\sigma_j^{-1} \in H$. Hence ηS is also transversal to H in G, and replacing S by ηS does not affect $i^*(\xi)$ by 12.4. Hence $\eta\, i^*(\xi)\eta^{-1} = i^*(\xi)$ for all $\xi \in G$, $\eta \in H$. It follows that $i^*(G) = i^*(H) = i^*(C) = C$. Hence $KC = G$, $K \cap C = 1$ and G is a split extension of K by C. Since m is prime to $|H|$, every element of H has the form ξ^m with $\xi \in H$ and (2) now shows that $H = CD$. But $D = [H, N] \leq G' \leq K$ since $G/K \cong C$ is Abelian. Since $K \cap C = 1$, this shows that the product $H = CD$ is direct, and that $G' \cap H = K \cap H = D$.

Finally, let $K_1 \triangleleft G$ and let G/K_1 be a p-group. Then $K_1H = G$ by 5.7 and so $G/K_1 \cong H/K_1 \cap H$ is Abelian. Hence K_1 contains G' and $|G : K_1| \leq |H : G' \cap H| = |C| = |G : K|$. Since $G/K \cap K_1$ is also a p-group, it follows that $K_1 \geq K$ and so G/K is the maximal p-quotient group of G.

<u>Corollary 12.51</u> If the group G has a Sylow p-subgroup H which is contained in the centre of its normalizer in G, then G has a normal $S_{p'}$-subgroup.

For in this case $C = H$ and G is the split extension of K by H. Hence K is a normal $S_{p'}$-subgroup of G.

(E) A soluble group whose Sylow subgroups are all Abelian is called an <u>A-group</u>. The following result is due to D.R. Taunt.

<u>Theorem 12.6</u> Let G be an A-group. Then
(i) $\mathfrak{z}G \cap G' = 1$; (ii) if S is any systemizer of G, then G is the split extension of G' by S; (iii) more generally, if T is a relative systemizer of K in G, where $K \triangleleft G$, then $K'T = G$ and $T \cap K' = 1$; (iv) the length of the derived series of G cannot exceed the number of distinct primes which divide $|G|$.

Proof : (i) Suppose if possible that $\zeta G \cap G'$ contains an element ξ of prime order p, let H be a Sylow p-subgroup of G containing ξ. Since H is Abelian, we can use the notation of 12·5. Then $\xi \in \zeta N \cap H = C$ and $\xi \in G' \cap H = D$. But $C \cap D = 1$ so $\xi = 1$, a contradiction. Hence $\zeta G \cap G' = 1$.

(ii) Let K/L be a chief factor of G with $K \leq G'$. Then G/L is a A-group by 5·6. Its derived group is G'/L. It follows from (i) that K/L is not a central factor of G. Hence $S \cap K \leq L$ by 10·4. Since this is true for all such K/L, we have $S \cap G' = 1$. By 10·42, SG' and this proves (ii).

(iii) Here K is an A-group by 5·6 and so $T \cap K' = 1$ by (ii); while by 10·8 (ii) and (v), $TK = G$. Since $K \leq TK'$ by 10·42, we have $TK' =$ and this proves (iii).

(iv) Suppose that $|G|$ has just n different prime factors. If $n = 1$, G is Abelian and $G' = 1$ by definition. Let $n > 1$ and let M be a minimal normal subgroup of G. If $|M| = p^m$, then every Sylow p-subg H contains M. Since H is Abelian, $H \leq C_G(M)$ and hence the automizer $G/C_G(M)$ is a p'-group. More generally, if K/L is any chief factor of G and if $|K : L| = p^l$, $C = C_G(K/L)$ is of index prime to p in G. Hence $|G/C|$ has at most $n-1$ different prime factors. By induction on n, we may assume that the $n-1$-st derived group of G/C is the unit subgroup. Since $(G/C)^{(n-1)} = CG^{(n-1)}/C$, this means that $G^{(n-1)} \leq C$. Thus $G^{(n-1)}$ centralizes every chief factor of G. Hence $G^{(n-1)}$ has a central series and so is nilpotent. But $G^{(n-1)}$ is also an A-group and so it is the direct product of Abelian groups viz. its Sylow subgroups. Hence $G^{(n-1)}$ is itself Abelian and $G^{(n)} = 1$, as required.

(F) Note in passing the following (forgotten)

Lemma 12.7 (i) In any group G, the Fitting subgroup is the intersection of the centralizers of the chief factors of G.

(ii) In any soluble group G, the hypercentre is the intersection of the systemizers of G.

Proof: (i) Let $F = \mathfrak{N}G$, and let $C = \cap\, C_G(H/K)$ over all chief factors H/K of G. By 9.3 (i), we have $F \leq \mathfrak{N}(G \bmod K) \leq C_G(H/K)$ and so $F \leq C$. On the other hand, C has a central series viz. the part of any chief series of G from 1 to C. Hence C is nilpotent; and so $C \leq F$ by 7.61, since $C \lhd G$. Thus $C = F$.

(ii) Let $H = \mathfrak{z}^{\infty}G$ and let K be the intersection of the systemizers of G. If L/M is any chief factor of G with $L \leq K$, then $L \leq MS$ for any systemizer S and so, by 10.4, L/M is a central factor of G. Hence $[L, G] \leq M$ and so $K \leq H$. Conversely, if L/M is any chief factor of G this time with $L \leq H$, then L/M is a central factor of G and so by 10.4 again $L \leq MS$. This is true for all such L/M and so $H \leq S$. This is true for all systemizers S of G and so $H \leq K$. Hence $H = K$.

(G) Let C_n denote a cyclic group of order n and let $\Phi_n = \mathrm{Aut}\, C_n$. If $C_n = \{\xi\}$ and $\alpha \in \Phi_n$, then $C_n = \{\xi^{\alpha}\}$ and so $\xi^{\alpha} = \xi^{a}$ where $(a, n) = 1$, by 2.5. Conversely, if $(a, n) = 1$, then $\xi^{s} \to \xi^{as}$ $(s = 0, 1, \dots, n-1)$ is an automorphism of C_n. If $\beta \in \Phi_n$ and $\xi^{\beta} = \xi^{b}$, then $\xi^{\alpha\beta} = \xi^{ab} = \xi^{\beta\alpha}$. Hence Φ_n is Abelian. The integer a is determined by the automorphism α only modulo n. Thus <u>Φ_n is isomorphic with the multiplicative group of prime residue classes mod n</u>, i.e. of all those residue classes (a) mod n for which $(a, n) = 1$.

The number $\varphi(n) = |\Phi_n|$ is called <u>Euler's function</u>. If $(m, n) = 1$, then Φ_{mn} is the direct product of subgroups isomorphic with Φ_m and Φ_n respectively by 8.2. Thus, if $(m, n) = 1$,

$$\Phi_{mn} \cong \Phi_m \times \Phi_n \quad \text{and} \quad \varphi(mn) = \varphi(m)\varphi(n)$$

Here we merely need to record

<u>Lemma 12.8</u> (i) The group of automorphisms of a cyclic group of order n is ~~the~~ an Abelian group of order $\varphi(n)$.

(ii) If p is a prime, $\varphi(p^m) = p^{m-1}(p-1)$.

(iii) Let H be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If H is cyclic, then G has a normal $S_{p'}$-subgroup.

Proof: (i) has already been shown and (ii) is clear since $C_{p^m} = \{\xi\}$ has only one maximal subgroup viz. $\{\xi^p\}$ of order p^{m-1}.

(iii) Since H is Abelian, we have $H \leq C_G(H) \leq N_G(H) = N$ and so the automizer $A_G(H) \cong N_G(H)/C_G(H)$ has order prime to p. Any prime q which divides $|A_G(H)|$ is therefore greater than p, by definition of p. But $q \leq p$ by (ii), so no such prime exists. Hence $H \leq \mathfrak{z}N$ and the result follows now from 12.51.

(H) A group G is called a <u>Z-group</u> if all its Sylow subgroups are cyclic. A group G is called <u>supersoluble</u> if all its chief factors are cyclic.

<u>Theorem 12.9</u>. (i) Z-groups are supersoluble.

(ii) If G is supersoluble, then G' is nilpotent; every maximal subgroup of G has index a prime; and if ϖ consists of all primes $\geq p$, ~~then~~ G has a normal S_ϖ-subgroup. Here p is any prime.

(iii) If G is a Z-group, then G' is cyclic and so are the systemizers S of G; moreover $|G'|$ and $|S|$ are coprime and G is the split extension of G' by S. Further, for each divisor d of $|G|$, G contains one and only one class of conjugate subgroups of order d.

Proof: (i) Let G be a Z-group. By 12.8, G has a normal $S_{p'}$-subgroup G_1 where p is the smallest prime dividing $|G|$. By induction on $|G|$, we may assume G_1 soluble. G/G_1 is a p-group and therefore soluble. Hence G is soluble. By 5.6 all sections of Z-groups are Z-groups. In particular, a chief factor of G must be cyclic of prime order, since it is an elementary Abelian prime-power group by the solubility of G.

(ii) Let G be supersoluble. By 12·7 (i), $F = \mathfrak{N}G = \cap\, C_G(H/K)$ over all chief factors H/K of G. By definition, each H/K is cyclic and so, by 12·8 (i), $G/C_G(H/K)$ is Abelian. All these centralizers contain G' by 7 Hence $G' \leq F$ and so G' is nilpotent.

By 9·2, if M is any maximal subgroup of a soluble group G, then $|G:M| = |H:K|$ for some chief factor H/K of G. If G is supersoluble, $|H:K|$ is a prime and so every maximal subgroup of G is of index a prime.

Let $G = G_0 > G_1 > \cdots > G_n = 1$ be a chief series of G. Since G is supersoluble, each $|G_{i-1} : G_i| = p_i$ is a prime. Choose a chief series for which $\sum_{i=1}^{n} i p_i = s$ is a maximum and suppose if possible that $p_i > p_{i+1}$ for some i. A Sylow p_i-subgroup G_i^*/G_{i+1} of G_{i-1}/G_{i+1} is normal in G_{i-1}/G_{i+1} by 5·4 (iv). G_i^*/G_{i+1} is therefore characteristic in G_{i-1}/G_{i+1}, he $G_i^* \lhd G$ and we may obtain a new chief series with larger s by replaci G_i by G_i^*. This contradicts our choice of chief series. Hence $p_i \leq p_{i+1}$ for all $i = 1, 2, \cdots, n-1$ and one of the terms G_i will be an $S_{\varpi'}$-subg of G with the given set of primes ϖ.

(iii) Let G be a Z-group. By (i) and (ii), G' is nilpotent. G' is al a Z-group, hence G' is cyclic by 8·2 (i) and (iii). Since G is an A-gr we have $G = G'S$, $G' \cap S = 1$, by 12·6 (ii); and so $S \cong G/G'$ is Abel Since S is also a Z-group, S is cyclic by 8·2 again.

Let q be the greatest prime dividing $|G|$. By (i) and (ii) the Sylow q-subgroup of G is normal in G. It is also cyclic. Hence G ha one and only one subgroup Q of order q. If q divides d, every subgrou H of G of order d must contain Q. By induction on $|G|$, we may assume that the Z-group G/Q has one and only one class of conjugate subgroups of order d/q. The result for G now follows.

If q does not divide d, and if H is any subgroup of G of order d then $Q \cap H = 1$ and QH/Q is a subgroup of order d of G/Q, and so H is an $S_{q'}$-subgroup of QH. By induction, we may assume that

~~subgroups~~ G/Q has a single class of conjugate subgroups K/Q of order $d_{\mathfrak{z}}$. By 9·5, each such K has a single class of conjugate subgroups H of order d and $K = HQ$. Hence in this case also the corresponding result follows for G.

This completes the proof of 12·9. Perhaps we should add

Lemma 12·91 Let A be a maximal normal Abelian subgroup of the supersoluble group G. Then $A = C_G(A)$.

Proof: Let $C = C_G(A)$. Since A is Abelian, we have $A \leq C$. Since $A \triangleleft G$, we have $C \triangleleft G$. Hence if $A < C$, there is a chief factor B/A of G such that $B \leq C$. Since G is supersoluble, B/A is cyclic of order a prime. Since $B \leq C_G(A)$, we have $A \leq \mathfrak{z}B$, and so $B/\mathfrak{z}B$ is cyclic. Hence $B = \mathfrak{z}B$ is Abelian by 7·2(iii), contrary to the definition of A. This contradiction shows that $A = C$.

Note that nilpotent groups are supersoluble. Hence 12·91 applies when G is a p-group.

Corollary 12·92 Let H be a Sylow p-subgroup of any group G and let A be a maximal normal Abelian subgroup of H. Then A is a Sylow p-subgroup of $C = C_G(A)$.

Proof: Since $A \triangleleft H$, we have $H \leq N = N_G(A)$. H is therefore a Sylow p-subgroup of N. Since $C \triangleleft N$, it follows that $H \cap C$ is a Sylow p-subgroup of C by 5·6. But $H \cap C = A$ by 12·91.