This picture represents a general case.

<u>Lemma 6·2</u> Let K be a subgroup of G containing H. If H sbn G then H sbn K. Similarly with pn, dsn, abn and $\lhd$ in place of sbn.

This is clear except for sbn, in which case we may state the stronger result:

<u>Lemma 6·3</u> (i) Let H sbn $K \leq G$ and let L be any subgroup of G. Then $L \cap H$ sbn $L \cap K$.

(ii) If H sbn K and K sbn L, then H sbn L.

(iii) If $H = \bigcap_{\lambda \in \Lambda} H_\lambda$ and each H_λ sbn G, then H sbn G.

Proof: (i) We may assume $H = H_0 \lhd H_1 \lhd \cdots \lhd H_r = K$. Then $L \cap H_{i-1} \lhd L \cap H_i$ for each $i = 1, 2, \ldots, r$ by 3·4 and so $L \cap H$ sbn $L \cap K$. (ii) is clear. (iii) It is sufficient to show that H sbn G and K sbn G implies $H \cap K$ sbn G. By (i), $H \cap K$ sbn $H \cap G = H$ and so this follows from (ii).

(B). Suppose that $J = \{H, H^\xi\}$ is a p-group, $\xi \in G$ and ~~H sbn G. By 6.2, H sbn J.~~ H pn G. By 6·2, H pn J. By 5·2 (iv), H sbn J. Hence $H \lhd J$ by 6·1; and so $H = H^\xi$ since H pn J.

Conversely, let H be a p-subgroup of G. By 5·4 (iii), H is contained in some Sylow p-subgroup S of G. Suppose that no other conjugate of H in G is contained in S. Let $J = \{H, H^\xi\}$, $\xi \in G$. By 5·4 (iii), $H \leq T \leq S_1$, where T is a Sylow p-subgroup of J and S_1 is a Sylow p-subgroup of G. Also by 5·4 (iii) and (ii), we have $H^{\xi\eta} \leq T$ for some $\eta \in J$. Since S and S_1 are conjugate in G, by 5·4 (ii), they contain the same number of conjugates of H in G. By hypothesis, this number is one. Hence $H^{\xi\eta} = H$ and H is conjugate to H^ξ in J. Since ξ was arbitrary, H pn G.

Thus we have proved

Theorem 6.4 Let H be a p-subgroup of G. Then H prn G if and only if each Sylow p-subgroup P of G contains exactly one of the conjugates of H in G.

Choosing P to contain H, this situation is expressed by saying that H is weakly closed in P with respect to G. This is the Wielandt terminology; but here we shall simply say H is pronormal in G.

Corollary 6.41 If $H \leq P$ where P is a Sylow p-subgroup of G and if H prn G, then $N_G(Q) \leq N_G(H)$ for every subgroup Q such that $H \leq Q \leq P$.

Corollary 6.42 If H_1 and H_2 are pronormal p-subgroups of G contained in the same Sylow p-subgroup P of G, then ~~ξ~~ $H_1 H_2$ is pronormal in G.

Note that H_1 and H_2 are normal in P by 6.41. If $(H_1 H_2)^{\xi} \leq P$, for some $\xi \in G$ then $H_i^{\xi} \leq P$ and so $H_i^{\xi} = H_i$ $(i = 1, 2)$ since H_i prn G. So $(H_1 H_2)^{\xi} = H_1 H_2$. So $H_1 H_2$ prn G.

It follows from 6.42 that if Q is any p-subgroup of G, there is a uniquely determined ~~maximal~~ pronormal subgroup Q_0 of G, which is maximal subject to being contained in Q. If this is H, then $H < K \leq Q$ implies that K is not pronormal in G. By 6.41, $H \leq \bigcap_{\xi \in N_G(Q)} Q^{\xi}$. The pronormal subgroups of G contained in a given Sylow p-subgroup P of G form a lattice $\mathcal{P} = \mathrm{Prn}_G(P)$. But this is not a sublattice of the lattice of all subgroups of P in general. For given H_1, H_2 in $\mathcal{P}$, although $H_1 H_2 \in \mathcal{P}$ by 6.42, in general $H_1 \cap H_2$ does not belong to $\mathcal{P}$, i.e. $\underline{H_1 \cap H_2}$ is usually smaller than $H_1 \cap H_2$ and even smaller than $[H_1, H_2]$. For example, if G is the octahedral group and $p = 2$, then P is an octic group. The subgroups of order 4 in G are all pronormal, but none of the subgroups of order 2 is pronormal. So $\mathcal{P}$ consists here of P itself, 1 and the three subgroups of order 4 in P.

Corollary 6.43 If $K \triangleleft G$, then any Sylow subgroup of K is pronormal in G.

For a Sylow p-subgroup P of G contains only one Sylow p-subgroup of K, viz. $P \cap K$.

Those pronormal subgroups of G which have the form $P \cap K$ with $K \triangleleft G$ form a ~~special~~ sublattice $\mathcal{P}_0$ of $\mathcal{P}$. Unlike $\mathcal{P}$, $\mathcal{P}_0$ is also a sublattice of the lattice of all subgroups of P. For if K_1 and K_2 are normal in G, so is $K_1 \cap K_2$ and $P \cap (K_1 \cap K_2) = (P \cap K_1) \cap (P \cap K_2)$. Further, $P \cap K_1 K_2 = (P \cap K_1)(P \cap K_2)$.

(C) The pronormal subgroups of a group G can also be characterized by their relation to the transitive permutational representations of G.

Suppose that G is represented transitively by permutations of a set X. Let H be a subgroup of G and let $N = N_G(H)$. Let Y be the set of all $y \in X$ which are invariant under H. If $y \in Y$ and $\xi \in N$, we have $yH = y$ and so $y\xi H = yH\xi = y\xi$. Hence $y\xi \in Y$. Thus N leaves Y invariant: $YN = Y$.

Let S_y be the stabilizer of y in G. If y and z are in Y, we have $y\eta = z$ for some $\eta \in G$ since G permutes X transitively. Since $H \leq S_y \cap S_z$, we have $H^\eta \leq S_y^\eta = S_z$. If H pron G, it is conjugate to H^η in $\{H, H^\eta\}$ and hence also in S_z. So $H^{\eta\zeta} = H$ for some $\zeta \in S_z$. So $\eta\zeta \in N$ and $y\eta\zeta = z$. Thus N permutes Y transitively whenever H is pronormal in G.

Conversely, suppose that in any transitive representation of G, N permutes transitively the symbols left invariant by H. Let $\xi \in G$ and let $J = \{H, H^\xi\}$. Then $JH = J$ and $JH^\xi = J$ or $J\xi^{-1}H = J\xi^{-1}$. Hence, in the transitive representation τ_J of G, H leaves fixed the cosets J and $J\xi^{-1}$. By hypothesis, $J = J\xi^{-1}\eta$ for some $\eta \in N$. Then $\xi^{-1}\eta \in J$ and transforms H^ξ into $H^\eta = H$. This is true for all $\xi \in G$. Hence H pron G.

Thus we have proved the first part of

<u>Theorem 6.6</u> Let H be a subgroup of G and let $N = N_G(H)$. Then (i) H pron G if and only if, in every transitive representation of G, N permutes transitively the symbols left invariant by H.

(ii) H abn G if and only if, in every transitive representation of G, H leaves at most one of the symbols invariant.

To prove (ii), let H abn G and let G permute X transitively. Then by (i) N permutes transitively the set Y of elements of X which are left invariant by H. But $H = N$. Hence $|Y| \leq 1$.

Conversely, if $|Y| \leq 1$ for every transitive representation of G, then H pron G by (i). If we take the representation of G to be τ_H, then $|Y| = |N : H|$ by 5.3 and so $N = H$ is dionormal in G. Hence H abn G.

By 4.6, every transitive representation of G is equivalent to τ_K for some subgroup K of G. In τ_K, H leaves the coset $K\xi$ invariant if and only if $H \leq K^{\xi}$. Thus $|Y| = 0$ unless H is contained in some conjugate of K. We may therefore restate the criterion for abnormal subgroups in the following two forms:

<u>Corollary 6.61</u> Let H be a subgroup of G. Then the following conditions are equivalent.

(i) H abn G.

(ii) For any subgroup K of G and any $\xi \in G$, $H \leq K \cap K^{\xi}$ implies $\xi \in K$.

(iii) Every subgroup of G containing H is dionormal in G, and H is not contained in the intersection of any two distinct conjugate subgroups of G.

We note also the following

<u>Lemma 6.7</u> (i) If H pron G and $K \triangleleft G$, then HK pron G.

(ii) If $K \triangleleft G$ and $K \leq H$, then H pron G if and only if H/K pron G/K.

~~(iii) If $K \triangleleft G$, H pron HK and HK pron G, then H pron G.~~

~~(i) and (ii) are clear. To prove (iii), let $\xi \in G$ and $J = \{H, H^{\xi}\}$. Since HK pron G, there is an element $\eta\xi \in JK$, where $\eta \in J$ and $\zeta \in K$ such that~~ ~~Sorry!~~

(D) We note next some easy corollaries of 6·6.

Lemma 6·61 The normalizers of pronormal subgroups are abnormal. In particular, normalizers of Sylow subgroups are abnormal.

Lemma 6·62 Any subgroup containing an abnormal subgroup is abnormal.

Lemma 6·63 Let H prn G and let $N = N_G(H)$. If two elements of the centre of H are conjugate in G, they are conjugate in N. If two normal subgroups of H are conjugate in G, they are conjugate in N.

Lemma 6·64 Let H abn G. Then no two distinct elements of the centre of H can be conjugate in G; and no two distinct normal subgroups of H can be conjugate in G.

Lemma 6·65 Let H prn G, let $N = N_G(H)$ and let $K = \{H^G\}$ be the normal closure of H in G. Then $KN = G$. Further, if $H \leq L$ sbn G, then $K \leq L$, and so $LN = G$.

Proof of 6·65. Let $\xi \in G$. Then $J = \{H, H^\xi\} \leq K$ and so $H^{\xi\eta} = H$ for some $\eta \in K$. Hence $\xi = (\xi\eta)\eta^{-1} \in NK = KN$, and so $KN = G$.

Let $L = L_0 \triangleleft L_1 \triangleleft \cdots \triangleleft L_r = G$. Since $H \leq L_{r-1} \triangleleft G$, we have $K \leq L_{r-1}$.

Omission Since H prn G, we have $K = \{H^K\}$ and hence $H \leq L_{r-2} \triangleleft L_{r-1}$ gives $K \leq L_{r-2}$ and so on.

By 6·3(iii), given any subgroup H of G, there is a uniquely determined smallest subnormal subgroup of G which contains H; this is the subnormal closure of H in G. By 6·65, <u>the subnormal closure of a pronormal subgroup coincides with its normal closure</u>.

Lemma 6·66 A pronormal subgroup cannot be permutable with any of its conjugates (other than itself).

For let H prn G and $H^\xi \neq H$, $\xi \in G$. Then we can choose $\xi \in J = \{H, H^\xi\}$. But $HH^\xi = H^\xi H$ would imply $J = HH^\xi$ by 2·7, contrary to 9·6(ii), sorry!

Lemma 6·67 Let H prn G and let $H \leq K \leq N = N_G(H)$. Then $N_G(K) \leq N$; and K prn G if and only if K/H prn N/H.

Suppose $K^\xi = K$, $\xi \in G$. Since $K \leq N$, we have $H^\xi \triangleleft K$. But H^ξ is conjugate to H in $J = \{H, H^\xi\}$ which is contained in K. Hence $H^\xi = H$ and $\xi \in N$.

If K pron G, then K pron N by 6.2 and so K/H pron N/H. Conversely, if K/H pron N/H, then K pron N. For any $\eta \in G$, there is an element ξ in $\{H, H^\eta\} \le J = \{K, K^\eta\}$ such that $H^{\eta\xi} = H$, since H pron G. Then $H \lhd K^{\eta\xi}$ and so $K^{\eta\xi} \le N$. Since K pron N, we have $K^{\eta\xi\zeta} = K$ for some $\zeta \in J_1 = \{K, K^{\eta\xi}\}$. But $\xi \in J$ and so $J_1 \le J$ and hence $\xi\zeta \in J$. Thus K and K^η are conjugate in J. This holds for all $\eta \in G$ and so K pron G.

<u>Lemma 6.68</u> Let $K \lhd G$ and let H/K be a pronormal p-subgroup of G/K. Then any Sylow p-subgroup P of H is pronormal in G.

Proof: Since H/K is a p-group, we have $H = KP$. Let $\xi \in G$ and let $J = \{P, P^\xi\}$. Then $\{H, H^\xi\} = JK$ and since H pron G, we have $H^{\xi\eta\zeta} = H$ for some $\eta \in J$, $\zeta \in K$. Then P and $P^{\xi\eta}$ are Sylow p-subgroups of $H = H^{\xi\eta}$ and hence they are conjugate in their join $J_1 = \{P, P^{\xi\eta}\}$. But $J_1 \le J$ since $\eta \in J$. Hence P and P^ξ are conjugate in J. This holds for all $\xi \in G$ and so P pron G.

This lemma shows that every pronormal p-subgroup of G/K is the image in the natural epimorphism of G onto G/K of some pronormal p-subgroup of G.

<u>Lemma 6.69</u> Let $G = LM$ where M is a normal p-subgroup of G and $L \cap M = 1$. Let P be a Sylow p-subgroup of L, $P_1 = [M, P]$ and $H = PP_1$. Then H is pronormal in G.

Proof: $Q = PM$ is a Sylow p-subgroup of G. Let $N = N_L(P)$. Since G and G/M have the same number of Sylow p-subgroups, we have $N_G(Q) = NM$. Let $H^\xi \le Q$. Then $Q^\xi = P^\xi M$ since $M \lhd G$ and $P^\xi \le Q$, $P^\xi \cap M = 1$. Hence $Q^\xi = Q$ and $\xi = \eta\zeta$ with $\eta \in N$, $\zeta \in M$. But $P^\eta = P$ and so $P_1^\eta = P_1$ and hence $H^\eta = H$. Thus $H^\xi = H^\zeta$. But $P_1 \lhd M$ by 7.1(iii) and $P^\zeta \le P[P, M] = H$. Hence $H^\zeta = H$, and so Q contains only one conjugate of H in G. So H pron G as stated.

(E). Theorem 6·8 The following conditions on a group G are equivalent

(i) G is nilpotent.

(ii) No proper subgroup of G is dionormal in G.

(iii) Every subgroup of G is subnormal in G.

(iv) Every pronormal subgroup of G is normal in G.

(v) Every maximal subgroup of G is normal in G and therefore contains G'.

(vi) Every Sylow subgroup of G is normal in G.

Proof: (i) $\Rightarrow$ (ii). For let H be a proper subgroup of G and let $1 = G_0 < G_1 < \cdots < G_r = G$ be a central series of G. Let G_i be the first term in this series which does not contain H, let $\xi \in H$ and $\eta \in G_i$. Since G_i/G_{i-1} is in the centre of G/G_{i-1}, we have $G_{i-1}\xi\eta = G_{i-1}\eta\xi$. Hence $G_{i-1}\eta\xi\eta^{-1} = G_{i-1}\xi \leq H$, since $G_{i-1} \leq H$ by definition of i. Therefore $\eta\xi\eta^{-1} \in H$. This holds for all $\xi \in H$ and $\eta \in G_i$. Hence $G_i \leq N_G(H)$. Since $G_i \nleq H$, we conclude that H is not dionormal in G.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv) follows from 6.1.

(iv) $\Rightarrow$ (v). For by 6.7, a maximal subgroup is either normal or abnormal, and in the second case it is pronormal. Alternatively, (ii) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (vi). For by 6·61 and 6·62 a maximal subgroup which contains the normalizer of a non-normal Sylow subgroup is abnormal and therefore not normal. Alternatively, (iv) $\Rightarrow$ (vi) since Sylow subgroups are pronormal.

(vi) $\Rightarrow$ (i). By induction on $|G|$. We need the following easy

Lemma 6·9. Let $H \triangleleft G$, $K \triangleleft G$ and $H \cap K = 1$. Then every element of H commutes with every element of K.

For let $\xi \in H$, $\eta \in K$ and $\zeta = \xi^{-1}\eta^{-1}\xi\eta$. Then $\xi^{-1}\eta^{-1}\xi \in K$ and so $\zeta \in K$, while $\eta^{-1}\xi\eta \in H$ and so $\zeta \in H$. Since $H \cap K = 1$, we have $\zeta = 1$ and $\xi\eta = \eta\xi$.

Suppose now that $G \neq 1$ and let S be a Sylow p-subgroup of G, $S \neq 1$ and $Z = \zeta S$. Then $Z \neq 1$ by 5.2(ii). Let T be a Sylow q-subgroup of G with $q \neq p$. Then $Z \cap T = 1$. By hypothesis $T \lhd G$. Also $S \lhd G$ and Z char S, so $Z \lhd G$ by 4.2. Hence T centralizes Z by 6.9. Also S centralizes $Z = \zeta S$. Thus $C = C_G(Z)$ contains a Sylow q-subgroup of G for all primes q, including $q = p$. So $C = G$ and $Z \leq \zeta G$. By 5.6, ZT/Z is a Sylow q-subgroup of G/Z. Since $Z \lhd G$ and $T \lhd G$, we have $ZT \lhd G$ and so $ZT/Z \lhd G/Z$. Hence the Sylow subgroups of G/Z are all normal, by 5.4(ii). By the induction hypothesis, G/Z has a central series with terms G_i/Z where $Z = G_1 < G_2 < \cdots < G_r = G$. Since $Z \leq \zeta G$, $1 = G_0 < Z = G_1 < \cdots < G_r = G$ is a central series of G. Thus G is nilpotent.