Upper & Lower Central Series

§7 ~~[illegible]~~

(A). Let ξ and η be elements of a group G. We define
$$[\xi, \eta] = \xi^{-1}\eta^{-1}\xi\eta = \xi^{-1}\xi^{\eta}$$
to be the <u>commutator</u> of ξ with η. Note that
$$[\eta, \xi] = [\xi, \eta]^{-1}$$
and that $[\xi,\eta] = 1$ if and only if ξ and η commute.

Let H and K be subgroups of G. We define
$$[H, K] = \{ [\xi, \eta] ; \xi \in H, \eta \in K \}.$$
Hence $[H, K] = [K, H]$ and $[H, K] = 1$ if and only if every element of H commutes with every element of K.

The group
$$G' = [G, G]$$
is called the (first) <u>derived group</u> of G. The second, third, ... derived groups of G are $G'' = (G')'$, $G''' = (G'')'$ and so on.

Besides 6.9, we have the following easy results concerning commutators

<u>Lemma 7.1</u>

(i) $[\xi\eta, \zeta] = [\xi, \zeta]^{\eta}[\eta, \zeta]$.

(ii) $[\xi, \eta\zeta] = [\xi, \zeta][\xi, \eta]^{\zeta}$.

(iii) $[H, K] \lhd \{H, K\}$.

(iv) If G admits a group of operators Γ, then $[H,K]^{\alpha} = [H^{\alpha}, K^{\alpha}]$ for all $\alpha \in \Gamma$. In particular, if $H \lhd G$ and $K \lhd G$, then $[H, K] \lhd G$ and $[H, K] \leq H \cap K$.

(v) H normalizes K if and only if $[H, K] \leq K$.

(vi) A section L/M of G is a central factor of G, if and only if $[L, G] \leq M$.

(vii) $K \lhd G$ and G/K is Abelian if and only if $G' \leq K$.

(viii) Let H, K and L be normal subgroups of G. Then $[HK, L] = [H,L][K,L]$ and $[H, KL] = [H,K][H,L]$

Proof: (i) $\xi^\eta = \xi[\xi,\eta]$. Hence $(\xi\eta)^\zeta = \xi\eta[\xi\eta,\zeta]$ but also $\xi^\zeta\eta^\zeta =$ $\xi[\xi,\zeta]\eta[\eta,\zeta] = \xi\eta[\xi,\zeta]^\eta[\eta,\zeta]$ and comparing gives (i).

(ii) follows from (i) by inversion, using (1).

(iii) In (i), let $\xi, \eta \in H$ and $\zeta \in K$. Then $[\xi,\zeta]^\eta = [\xi\eta,\zeta][\eta,\zeta]^{-1}$ $\in [H,K]$. This is true for all such ξ, η, ζ. Hence $H \leq N_G([H,K])$. Similarly, in (ii), let $\xi \in H$ and $\eta, \zeta \in K$. We obtain $K \leq N_G([H,K])$. Hence $\{H,K\}$ normalizes $[H,K]$; and also contains it. So (iii) follows.

(iv) $[\xi,\eta]^\alpha = [\xi^\alpha, \eta^\alpha]$ for all ξ, η in G and $\alpha \in \Gamma$. Hence $[H,K]^\alpha = [H^\alpha, K^\alpha]$. If H and K are normal in G, the fact that $[H,K] \leq H \cap K$ follows from (v).

(v) Let $\xi \in H$, $\eta \in K$. If H normalizes K, then $[\xi,\eta] = \xi^{-1}\eta^{-1}\xi\cdot\eta$ lies in K, since $\xi^{-1}\eta^{-1}\xi \in K$ and $\eta \in K$. Hence $[H,K] \leq K$. Conversely, the latter relation implies that $\xi^{-1}\eta^{-1}\xi \in K$ for all $\xi \in H$, $\eta \in K$. So H normalizes K.

(vi) Let $\xi \in L$ and $\eta \in G$. If L/M is a central factor of G, then $M\xi$ commutes with $M\eta$, so $[M\xi, M\eta] = M$. But $M \lhd G$ and so $[M\xi, M\eta] = M[\xi,\eta]$ and $[\xi,\eta] \in M$. Thus $[L,G] \leq M$. Conversely, let $[L,G] \leq M$. Since $M \leq L$, it follows that $[M,G] \leq M$ Omission and so $M \lhd G$ by (v). Further, $[\xi,\eta] \in M$ and so $[M\xi, M\eta] = M$. Hence $L/M \leq \mathfrak{z}(G/M)$ and L/M is a central factor of G.

(vii) If $K \lhd G$ and G/K is Abelian, then G/K is a central factor of G and so $G' \leq K$ by (vi). Conversely, if $G' \leq K$, then $[K,G] \leq K$ and $K \lhd G$ by (v); and G/K is a central factor of G by (vi), hence G/K is Abelian.

(viii) Take ξ, η, ζ in (i) and (ii) to be arbitrary elements of H, K, L respectively and use (iv) which ensures the normality of $[H,L]$, $[K,L]$, ... in G

$$\zeta(G \bmod K) = L.$$

The <u>upper central series</u> of G consists of the groups $\zeta^n G$ defined inductively as follows:

$$\zeta^1 G = \zeta G, \quad \zeta^{n+1} G = \zeta(G \bmod \zeta^n G), \ (n = 1, 2, \cdots).$$

Also we write $\zeta^0 G = 1$. Hence

$$1 = \zeta^0 G \leq \zeta^1 G \leq \zeta^2 G \leq \cdots$$

and $\zeta^n G / \zeta^{n-1} G$ is the centre of $G/\zeta^{n-1}G$. ~~The~~ We have

$$\zeta^n G \text{ char } G$$

for all n. $\zeta^2 G$, $\zeta^3 G$, $\cdots$ are called the second, third, $\cdots$ centres of G. We define

$$\zeta^\infty G = \bigcup_{n=1}^{\infty} \zeta^n G$$

to be the <u>hypercentre</u> of G. If r is the least non-negative integer such that $\zeta^r G = \zeta^{r+1} G$, then

$$1 = \zeta^0 G < \zeta^1 G < \cdots < \zeta^r G = \zeta^\infty G.$$

Suppose that $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots$ and that each G_i/G_{i-1} is a central factor of G. Then we have an <u>ascending central series</u> of G. By induction on n, we obtain

$$G_n \leq \zeta^n G \qquad (n = 0, 1, 2, \cdots).$$

The upper central series is the uppermost ascending central series of G. Hence G is nilpotent if and only if $G = \zeta^\infty G = \zeta^r G$ for some $r \geq 0$. The least such r is called the <u>class</u> of G, following W.B. Fite. Abelian groups are the same as nilpotent groups of class 1.

<u>Lemma 7.2</u>. (i) If $H \triangleleft G$ and $H \cap \zeta^r G > H \cap \zeta^{r-1} G$ for some r, then $H \cap \zeta^s G > H \cap \zeta^{s-1} G$ for $s = 1, 2, \cdots, r$.

(ii) If G is non-Abelian, then $G/\zeta G$ is not cyclic.

(iii) If $|G| = p^n$, then the class of G is at most $n-1$, if $n > 1$.

Proof: (i) Let $\xi \in H \cap \zeta^r G$, $\xi \notin \zeta^{r-1} G$. Then there is an element η of G such that $[Z\xi, Z\eta] \neq Z$ where $Z = \zeta^{r-2} G$. Since $H \triangleleft G$, $\zeta = [\xi, \eta] \in H$. Also $\zeta \in \zeta^{r-1} G$, but $\zeta \notin Z = \zeta^{r-2} G$. Hence the result.

<u>Lemma 7.2</u>. (i) If $H \lhd G$ and $H \cap \mathfrak{z}^r G > H \cap \mathfrak{z}^{r-1} G$ for some $r \geq 1$, then $H \cap \mathfrak{z}^s G > H \cap \mathfrak{z}^{s-1} G$ for all $s = 1, 2, \dots, r$.

(ii) If $|G| = p^n$ and $|H| = p^r$ and $H \lhd G$, then $H \leq \mathfrak{z}^r G$.

(iii) If $G' \neq 1$, then $G/\mathfrak{z}G$ is not cyclic.

(iv) If $|G| = p^n$, then, for $n > 1$, the class of G is at most $n-1$. In particular, groups of order p^2 are Abelian.

Proof. (i) Let $\xi \in H \cap \mathfrak{z}^r G$, $\xi \notin H \cap \mathfrak{z}^{r-1} G$ and let $r > 1$, $Z = \mathfrak{z}^{r-2} G$. Then there is an element η of G such that $[Z\xi, Z\eta] \neq Z$. Since $H \lhd G$, $\zeta = [\xi, \eta] \in H$ by 7.1(v). Also $\zeta \in \mathfrak{z}^{r-1} G$ since $\xi \in \mathfrak{z}^r G$. But $Z\zeta \neq Z$ and so $\zeta \notin \mathfrak{z}^{r-2} G = Z$. Hence $H \cap \mathfrak{z}^{r-1} G > H \cap \mathfrak{z}^{r-2} G$.

(ii) follows from (i).

(iii) If $G/\mathfrak{z}G$ is cyclic, then $G = \{\mathfrak{z}G, \xi\}$ for some ξ and $[\xi, \zeta] = 1$ for all $\zeta \in \mathfrak{z}G$. Hence G is Abelian.

(iv) Let $G = \mathfrak{z}^c G > \mathfrak{z}^{c-1} G > \dots > \mathfrak{z}G > 1$ so that c is the class of G. If $c = 1$, the result follows from $n > 1$. If $c > 1$, then $G/\mathfrak{z}^{c-1}G$ is of order at least p^2 and so $|G| \geq p^{c+1}$.

(C) If X is any subset of the group G, we denote by X^G the set of all ξ^α with $\xi \in X$, $\alpha \in G$. Then $\{X^G\}$ is the least normal subgroup of G containing X. It is called the <u>normal closure</u> of X in G. Note that any subgroup of G which is generated by the union of a certain number of classes of conjugates in G is normal in G.

<u>Theorem 7.3</u> Let H be a subgroup of G and let $H_1, H_2, \dots, H_r$ be in any order the conjugates of H in G, so that $r = |G : N_G(H)|$. Then
$$\{H^G\} = H_1 H_2 \cdots H_r.$$

This theorem is due to Ito.

Proof. Let $K = \{H_1, H_2, \dots, H_r\}$ so that $K = \{H^G\}$. Every element $\xi \in K$ can be expressed in the form
$$\xi = \xi_1 \xi_2 \cdots \xi_s \qquad (1)$$
where $\xi_\lambda \in H_{i_\lambda}$, $\lambda = 1, 2, \dots, s$. We say that the expression (1) then

belongs to the ordered multiplet $i = (i_1, i_2, \dots, i_s)$. Choose (1) so that s is as small as possible; and among the possible expressions for ξ with minimum s, choose (1) so that i comes as early as possible in the lexicographic ordering of ordered multiplets. Suppose if possible that for some $\lambda < \mu$, we have $i_\lambda \geqslant i_\mu$. Then we have

$$\xi = \xi_1' \xi_2' \cdots \xi_s' \qquad (2)$$

where $\xi_\nu = \xi_\nu'$ for $\nu < \lambda$ and also for $\nu > \mu$; $\xi_\lambda' = \xi_\mu$; and for $\lambda < \alpha \leq \mu$ $\xi_\alpha' = \xi_\mu^{-1} \xi_{\alpha-1} \xi_\mu$. Then (2) belongs to the ordered multiplet $j = (i_1, \dots, i_{\lambda-1}, i_\mu, \dots)$ of length s; and j precedes i in lexicographic order since $i_\mu < i_\lambda$. This contradicts the definition of i. Hence $i_1 \leq i_2 \leq \dots \leq i_s$. But if $i_\lambda = i_{\lambda+1}$, the two neighbouring factors $\xi_\lambda, \xi_{\lambda+1}$ in (1) can be coalesced to a single factor which likewise belongs to $H_{i_\lambda} = H_{i_{\lambda+1}}$, ~~again~~ contrary to the minimality of s. Hence $i_1 < i_2 < \dots < i_s$ and so $\xi \in H_1 H_2 \cdots H_r$. Since ξ was any element of K, we thus obtain $K = H_1 H_2 \cdots H_r$ as stated.

(D). Lemma 7.4 Let $G = \{X\}$ and let $H = \{Y^G\}$. Then $[H, G] = \{Z^G\}$ where Z is the set of all commutators $[\eta, \xi]$ with $\eta \in Y$, $\xi \in X$.

Proof: By 7.1 (iv), $K = [H, G] \triangleleft G$; also $Z \leqslant K$. Hence $K_1 = \{Z^G\}$ is contained in $K \leqslant H$. Let $H_1 = \zeta(G \bmod K_1)$. Since $[\eta, \xi] \in Z \leqslant K_1$ for all $\eta \in Y$ and all $\xi \in X$, we have $[K_1\eta, K_1\xi] = K_1$. ~~Hence~~ Also G/K_1 is generated by the cosets $K_1\xi$, $\xi \in X$. Hence $K_1\eta \in \zeta(G/K_1)$ i.e. $\eta \in H_1$ for all $\eta \in Y$. But $H_1 \triangleleft G$ and so $H_1 \geqslant H = \{Y^G\}$. Thus H/K_1 is a central factor of G and so $K = [H, G] \leqslant K_1$ by 7.1 (vi). Since $K_1 \leqslant K$, we obtain $K = K_1$ as required.

We define inductively

$$[\xi_1, \xi_2, \dots, \xi_n] = [[\xi_1, \dots, \xi_{n-1}], \xi_n],$$
$$[H_1, H_2, \dots, H_n] = [[H_1, \dots, H_{n-1}], H_n],$$

for $n = 3, 4, \dots$, where the ξ's and H's are respectively elements and subgroups of G.

The groups $\gamma_n G$ defined inductively by

$$G = \gamma_1 G , \quad [\gamma_n G, G] = \gamma_{n+1}(G) , \quad n = 1, 2, \cdots$$

are characteristic subgroups of G, by 7·1 (iv) and the series

$$G = \gamma_1 G \geq G' = \gamma_2 G \geq \gamma_3 G \geq \cdots$$

is called the <u>lower central series</u> of G.

Suppose that $G = G_1 \trianglerighteq G_2 \trianglerighteq G_3 \trianglerighteq \cdots$ and that each G_i/G_{i+1} is a central factor of G. Then we have a <u>descending central series</u> of G. By induction on n, we obtain

$$G_n \geq \gamma_n G \qquad (n = 1, 2, 3, \cdots).$$

The lower central series of G is the lowermost descending central series of G.

G is nilpotent if and only if $\gamma_{r+1} G = 1$ for some $r \geq 0$ and the least such r is the class of G. If c is the class of G, then

$$\zeta^n G \geq \gamma_{c-n+1} G \qquad (n = 0, 1, \cdots, c).$$

<u>Lemma 7·5</u> (i) If $G = \{X\}$, then $\gamma_n G$ is generated by all the commutators $[\xi_1, \xi_2, \cdots, \xi_n]$ with $\xi_i \in X$ together with their conjugates in G.

(ii) If $H \triangleleft G$ and $\gamma_r G \leq H(\gamma_{r+1} G)$, then $\gamma_s G \leq H(\gamma_{s+1} G)$ for $s = r+1, r+2, \cdots$ and so $\gamma_r G \leq H(\gamma_{r+k} G)$ for all $k \geq 1$.

(iii) If G is nilpotent and H is any subgroup such that $HG' = G$, then $H = G$.

(iv) If H is a subgroup of the nilpotent group G, then $\gamma_n H \subseteq \gamma_n G$, $n = 1, 2, \ldots$. In particular, H is nilpotent.

Proof: (i) follows from 7·4 by induction on n.

(ii) By hypothesis, every element of $\gamma_r G$ has the form $\eta\zeta$ with $\eta \in H$ and $\zeta \in \gamma_{r+1} G$. Hence $\gamma_{r+1} G$ is generated by elements of the form $[\eta\zeta, \xi] = [\eta, \xi]^\zeta [\zeta, \xi]$ with $\xi \in G$. Hence $[\eta, \xi]^\zeta \in H$ since $H \triangleleft G$ and $[\zeta, \xi] \in K = \gamma_{r+2} G$. Since $K \triangleleft G$, we thus have $\gamma_{r+1} G \leq HK$.

(iii). This is a consequence of 6·8. If $H < G$, let M be a maximal subgroup of G containing H. Then by 6·8 (v), $M \triangleleft G$ and so G/M is cyclic of prime order, and $M \geq G'$ by 7·1 (vii). Hence $HG' \leq MG' = M < G$ contrary to hypothesis.

(iv) is obvious by induction on n.

(E) Let $H_i \lhd G$, $K_j \lhd G$, $\cdots$, $M_r \lhd G$ for $i=1,\cdots,h$; $j=1,\cdots,k$; $\cdots$; $r=1,\cdots,m$. Then 7·1 (viii) gives the general distributive law of commutation,

$$[\prod_{i=1}^{h} H_i, \prod_{j=1}^{k} K_j, \cdots, \prod_{r=1}^{m} M_r] = \prod_{i,j,\dots,r} [H_i, K_j, \cdots, M_r], \qquad (1)$$

by induction. The order of the factors is irrelevant since all the groups are norm

In particular, if $H \lhd G$ and $K \lhd G$, we obtain

$$\gamma_n(HK) = \prod [L_1, L_2, \cdots, L_n], \qquad (2)$$

where the product is taken over all 2^n terms having each L_i equal to either H or K. Suppose that in $L = [L_1, L_2, \cdots, L_n]$ just r terms are equal to H and s terms equal to K. By 7·1 (v) and induction, we then obtain $L \leq \gamma_r(H) \cap \gamma_s(K)$. Here we must interpret $\gamma_0(H) = \gamma_0(K) =$
. Consequently (2) gives

$$\gamma_n(HK) \leq \prod_{r=0}^{n} \left(\gamma_r(H) \cap \gamma_{n-r}(K)\right) \qquad (3)$$

Here again the $n+1$ factors are normal in G since e.g. $\gamma_r(H)$ char $H \lhd G$

Suppose now that H and K are nilpotent, of class a and b respecti
If we take $n = a+b+1$ in (3), we then have in each factor either $r \geq a+1$ or $n-r \geq b+1$. But $\gamma_{a+1} H = \gamma_{b+1} K = 1$. Hence $\gamma_{a+b+1}(HK) = 1$ and we obtain

<u>Theorem 7·6</u>. If H and K are normal subgroups of G and if H and K are nilpotent, of class a and b respectively, then HK is nilpotent of class at most $a+b$. (Fitting)

~~Note~~ The theorem that ~~[illegible]~~, the product of two normal nilpoten subgroups H and K is nilpotent follows also, as noted by Wendt, from 6·8 (vi). For if $L = HK$ and $M = H \cap K$, then $|L| = |H| \cdot |K : M|$ by 5·5 (or equally 3·4) and so $|L|_p \, |M|_p = |H|_p \, |K|_p$. If S and are the unique Sylow p-subgroups of H and K respectively, then S char H and so $S \lhd G$. Similarly $T \lhd G$ and so $ST \lhd G$. Now $|S \cap T| \leq |$
Hence $|ST| = |S| \cdot |T : S \cap T| \geq |H|_p \, |K|_p / |M|_p = |L|_p$. So $|ST| = |L$
and ST is a Sylow p-subgroup of L. Hence the Sylow subgroups of L are all normal and so L is nilpotent by 6·8 (vi).

Corollary 7.61. Every group G has a normal nilpotent subgroup $\mathfrak{N}G$ which contains every other normal nilpotent subgroup of G. $\mathfrak{N}G$ is called the Fitting subgroup of G. If H sbn G, then $\mathfrak{N}H = H \cap \mathfrak{N}G$. In partic every subnormal nilpotent subgroup of G is contained in the Fitting subgroup of G.

Proof: By 7.6, a normal nilpotent subgroup of maximal order in G contains every normal nilpotent subgroup of G and hence is unique. Clearly $\mathfrak{N}G$ char G. Hence, if $H \triangleleft G$, $\mathfrak{N}H \triangleleft G$ and so $\mathfrak{N}H \leq H \cap \mathfrak{N}G$. But $H \cap \mathfrak{N}G \triangleleft G$ and is nilpotent*, so $H \cap \mathfrak{N}G \leq \mathfrak{N}H$. Thus $\mathfrak{N}H = H \cap \mathfrak{N}$ if $H \triangleleft G$. This extends at once to the case of subnormal H. If H is itself nilpotent, then $H = \mathfrak{N}H$.

~~* Note that a subgroup L of a nilpotent group M of class c is nilpotent of class not exceeding c, since $\gamma_{c+1} L \leq \gamma_{c+1} M = 1$. This remark should have been inserted before.~~

(F) Let ξ, η, ζ be elements of any group G, let $\alpha = \xi\zeta\xi^{-1}\eta\xi$ and let β and γ be obtained from α by cyclic permutation of ξ, η, ζ. Then $[\xi, \eta^{-1}, \zeta]^{\eta} = \eta^{-1}(\eta\xi^{-1}\eta^{-1}\xi)\zeta^{-1}(\xi^{-1}\eta\xi\eta^{-1})\zeta\eta = \alpha^{-1}\beta$. Thus we obtain the first part of

<u>Theorem 7.7</u> (i) $[\xi, \eta^{-1}, \zeta]^{\eta}[\eta, \zeta^{-1}, \xi]^{\zeta}[\zeta, \xi^{-1}, \eta]^{\xi} = 1$.

(ii) Let H, K and L be subgroups of G. Then any normal subgroup M of G which contains two of the groups $U = [K, L, H]$, $V = [L, H, K]$ and $W = [H, K, L]$ also contains the third.

(iii) In particular, if H, K and L are normal in G, then $U \leq VW$, $V \leq WU$ $W \leq UV$.

(iv) If $L \leq \{H, K\}$ and if $[H, K, L] = 1$, then $[L, H, K] = [L, K, H] \lhd \{H,$

Proof: (ii). In (i), let $\xi \in H$, $\eta \in K$ and $\zeta \in L$, and suppose that M contains U and V. The three factors in (i) belong respectively to W^{η}, U^{ζ} and V^{ξ}. Since $M \lhd G$, $U^{\zeta} \leq M^{\zeta} = M$ and similarly $V^{\xi} \leq M$. Hence $[\xi, \eta^{-1}, \zeta] \in \eta M \eta^{-1} = M$, and so $M\zeta$ commutes with $M[\xi, \eta^{-1}]$ for all $\xi \in H$, $\eta \in K$. Therefore the centralizer of $M\zeta$ in G/M contains $M[H, K]/M$. This is true for all $\zeta \in L$. Hence $[H, K, L] = W \leq M$ as required. (iii) is a corollary of (ii), by 7.1 (iv).

(iv) Let C be the centralizer of $[H, K]$ in $J = \{H, K\}$. By 7.1 (iii), $C \lhd J$ and by hypothesis $L \leq C$. Hence $[L, H] \leq C$ by 7.1 (v). Let $\xi \in H$, $\eta \in K$ and $\tau \in [L, H]$. Then $\tau \in C$ and $[\xi, \eta^{-1}] \in [H, K]$ so that τ commutes with $[\xi, \eta^{-1}]$. But $\eta^{\xi} = [\xi, \eta^{-1}]\eta$ and so 7.1 (ii) gives $[\tau, \eta^{\xi}] = [\tau, \eta]$. Hence $[L, H, K^{\xi}] = [L, H, K]$. Since $\xi \in H$, we also have $[L, H] = [L, H]^{\xi}$ by 7.1 (iii). Hence $[L, H, K]^{\xi} = [[L, H]^{\xi}, K^{\xi}] = [L, H, K^{\xi}] = [L, H, K]$. Thus H normalizes $[L, H, K]$. By 7.1 (iii), K also normalizes $[L, H, K]$. So $[L, H, K] \lhd J$. But $[K, H, L] = [H, K, L] = 1$ by hypothesis. Interchanging H and K, we therefore find that $[L, K, H] \lhd J$ also. Since $[L, K, H] = [K, L, H]$, (iv) now follows from (iii).

(G) As a corollary of 7·7 (iii), we have

Theorem 7·8 (i) Let $H_0 \geqslant H_1 \geqslant H_2 \geqslant \cdots$ be a series of normal subgroups of G and let K be a subgroup of G such that $[H_i, K] \leqslant H_{i+1}$ for all $i = 0, 1, 2, \cdots$. Then $[H_i, \gamma_n K] \leqslant H_{i+n}$ for all $i \geqslant 0$ and $n \geqslant 1$.

(ii) In any group G, $[\gamma_l G, \gamma_m G] \leqslant \gamma_{l+m} G$ and, for $l \geqslant m$, $[\zeta^l G, \gamma_m G] \leqslant \zeta^{l-m} G$.

(iii) If $G^{(n)}$ is the n-th derived group of G, then $G^{(n)} \leqslant \gamma_{2^n} G$.

Proof: (i) by induction on n. For $n=1$, the theorem is true by hypothesis. Let $n > 1$. In 7·7 (i), take $H = \gamma_{n-1} K$, $L = H_i$ and $M = H_{i+n}$. Then $U = [K, H_i, \gamma_{n-1} K] = [H_i, K, \gamma_{n-1} K] \leqslant [H_{i+1}, \gamma_{n-1} K]$ by hypothesis and so $U \leqslant M$ by induction. On the other hand, $V = [H_i, \gamma_{n-1} K, K]$ $\leqslant [H_{i+n-1}, K]$ by induction and so $V \leqslant M$ by the hypothesis. Since $M \lhd G$ it follows that $W = [\gamma_{n-1} K, K, H_i] = [\gamma_n K, H_i] = [H_i, \gamma_n K] \leqslant M = H_{i+n}$, as required.

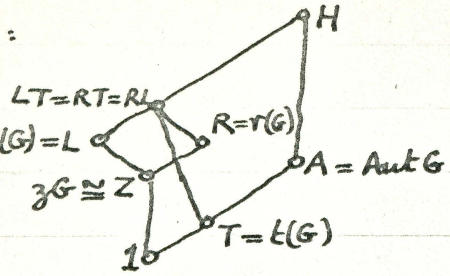
(ii) If each of the sections H_i / H_{i+1} is a central factor of G, we can take $K = G$ in (i). Applying this remark to the lower and upper central series of G gives (ii).

(iii) By induction on n, we may assume $G^{(n)} \leqslant \gamma_{2^n} G$ for some n, since $G^{(0)} = G = \gamma_1 G$. Then $G^{(n+1)} = [G^{(n)}, G^{(n)}] \leqslant [\gamma_{2^n} G, \gamma_{2^n} G] \leqslant \gamma_{2^{n+1}} G$ by (ii).

(H) If $\alpha \in A = \operatorname{Aut} G$ and $\xi \in G$, then $r(\xi^\alpha) = r(\xi)^\alpha = \alpha^{-1} r(\xi) \alpha$. Thus A normalizes $R = r(G)$. The group

$$H = AR$$

is called the holomorph of G. Note that H is a transitive subgroup of $\Sigma(G)$ and that A is the stabilizer of 1 in H. By 5·3, the centralizer of R in $\Sigma(G)$ is $L = l(G)$. Both R and L are isomorphic with G and $R \cap L = Z$ ~~consists~~ of all the permutations $r(\zeta) = l(\zeta)$ with $\zeta \in \zeta G$. The stabilizer of 1 in RL is $T = t(G)$ the group of inner automorphisms of G. Hence $A \cap RL = T$. The general picture is as shown here:



Given a group Γ of automorphisms of G, i.e. any subgroup of A, it is often convenient to think of Γ and G as subgroups of a larger group ΓG with $\Gamma \cap G = 1$, $G \lhd \Gamma G$, and $\xi^\alpha = \alpha^{-1}\xi\alpha$ for $\xi \in G$, $\alpha \in \Gamma$. Such a group can be obtained from $\Gamma . r(G)$ by identifying $r(\xi)$ with ξ. We call ΓG the <u>holomorphic extension</u> of G by Γ. This device gives a meaning to the commutators $[\xi, \alpha]$ with $\xi \in G$, $\alpha \in \Gamma$ viz. $\xi^{-1}\xi^\alpha$. Note that the device is purely a notational one.

<u>Theorem 7.9</u> Let Γ be a group of automorphisms of G and suppose that there exists a chain of subgroups $G = G_0 > G_1 > G_2 > \cdots > G_m = 1$ such that $[G_{i-1}, \Gamma] \leq G_i$ for each $i = 1, 2, \cdots, m$. Then:

(i) Every prime divisor p of $|\Gamma|$ also divides $|G_1|$.

(ii) Γ is nilpotent of class at most $\frac{1}{2}m(m-1)$.

(iii) $[G, \Gamma]$ is a normal nilpotent subgroup of G and $\Gamma[G, \Gamma]$ is a normal nilpotent subgroup of ΓG.

(iv) If each $G_i \lhd G$, then the class of Γ is at most $m-1$.

Proof: (i). By induction on m, since $\Gamma = 1$ when $m = 1$. Let α be any element of Γ of prime order p. If $[G_1, \alpha] \neq 1$, then p divides $|G_1|$ by induction. If $[G_1, \alpha] = 1$, there is an element $\xi \in G - G_1$ such that $\eta = [\xi, \alpha] \neq 1$, since otherwise $\alpha = 1$, which is not the case. Then $\eta \in G_1$ and so $\eta^\alpha = \eta$ by hypothesis. Also $\xi^\alpha = \xi\eta$, hence $\xi^{\alpha^r} = \xi\eta^r$ by induction on $r = 1, 2, \ldots$. Since $\alpha^p = 1$, it follows that $\eta^p = 1$. Hence p divides $|G_1|$ in this case also.

(ii) Define $G_0^* = G$, $G_{i+1}^* = [G_i^*, \Gamma]$, $i = 0, 1, 2, \ldots$. Then by induction on i, we have $G_i^* \leq G_i$ for all i. By 7.1 (iii), $G_{i+1}^* \lhd \{G_i^*, \Gamma\}$.

Hence there is no loss of generality in assuming that $G_{i+1} \lhd \{G_i, \Gamma\}$ for $i = 0, 1, \dots, m-1$. Since $m = 1$ implies $\Gamma = 1$, we use induction on m and suppose $m > 1$. Hence if $l = \frac{1}{2}(m-1)(m-2)$, we may assume that $\gamma_{l+1}\Gamma$ centralizes G_1. Write $\Gamma_r = \gamma_r\Gamma$, so that $[G, \Gamma, \Gamma_r] = 1$ for all $r > l$; and therefore, by 7·7 (iv), $[\Gamma_r, G, \Gamma] = [\Gamma_r, \Gamma, G] = [\Gamma_{r+1}, G]$ for all $r > l$. Hence $[\Gamma_{l+m}, G] = [\Gamma_{l+m-1}, G, \Gamma] = [\Gamma_{l+m-2}, G, \Gamma, \Gamma] = \cdots$

$$= [\Gamma_{l+1}, G, \underbrace{\Gamma, \Gamma, \dots, \Gamma}_{m-1}] \leq [G_1, \underbrace{\Gamma, \Gamma, \dots, \Gamma}_{m-1}] \leq [G_2, \underbrace{\Gamma, \dots, \Gamma}_{m-2}] \leq G_m = 1$$

Thus Γ_{l+m} centralizes G and so $\Gamma_{l+m} = 1$ and Γ is nilpotent of class at most $l + m - 1 = \frac{1}{2}m(m-1)$ as required.

(iii). By 7·1 (iii), $[G, \Gamma] \lhd G$. Hence $\overline{\Gamma} = \Gamma[G, \Gamma]$ is the normal closure of Γ in ΓG. Taking $G_1 = [G, \Gamma]$, $G_2 = [G_1, \Gamma]$, $\dots$ we find that ΓG_{i+1} is the normal closure of Γ in ΓG_i. Thus we have

$$\Gamma = \Gamma G_m \lhd \Gamma G_{m-1} \lhd \cdots \lhd \Gamma G_1 = \overline{\Gamma} \lhd \Gamma G.$$

Hence Γ sbn ΓG. By (ii), Γ is nilpotent. Hence by 7·61, $\Gamma \leq \mathfrak{N}(\Gamma G)$ the Fitting subgroup of ΓG. Hence $\overline{\Gamma} \leq \mathfrak{N}(\Gamma G)$ and so $\overline{\Gamma}$ is nilpotent. A fortiori, $[G, \Gamma]$ is nilpotent.

(iv) is a corollary of 7·8 (i).